Solutions : Assignment 4

① $V = \frac{\tau}{2}\int d^3x\ (\nabla\varphi)^2$

$$T = \frac{\rho}{2}\int d^3x\ \left(\frac{\partial\varphi}{\partial t}\right)^2$$

$L = T - V$, equation of motion is obtained by minimizing $S = \int dt\, L$

$$S = \int dt\, d^3x \left[\frac{\tau}{2}(\nabla\varphi)^2 - \left(\frac{\partial\varphi}{\partial t}\right)^2 \cdot \frac{\rho}{2}\right]$$

$$\delta S = \int dt\, d^3x \left[\frac{\tau}{2}\cdot 2\cdot \nabla\varphi\ \delta(\nabla\varphi) - \frac{\rho}{2}\cdot 2\dot{\varphi}\,(\delta\dot{\varphi})\right]$$

Interchanging δ & derivatives,
Integrating by parts & ignoring the surface terms we get

$$\delta S = \int dt\, d^3x \left[\tau\nabla^2\varphi - \rho\ddot{\varphi}\right]\delta\varphi = 0$$

Since $\delta\varphi$ is arbitrary,

$$\tau\nabla^2\varphi = \rho\ddot{\varphi}\ ;$$

or $\nabla^2\varphi = \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2}$ where $\frac{1}{v^2} = \frac{\rho}{\tau}$ & $\ddot{\varphi} = \frac{\partial^2\varphi}{\partial t^2}$

Q2

$$Z[J] = e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x)\,\Delta(x-y)\,J(y)}$$

$$\frac{\delta Z}{\delta J(z_0)} = e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x)\,\Delta(x-y)\,J(y)} * \Big[-\frac{1}{2}\int d^4x\, d^4y\, \delta(x-z_0)\,\Delta(x-y)\,J(y)$$

$$-\frac{1}{2}\int d^4x\, d^4y\, J(x)\,\Delta(x-y)\,\delta(y-z_0)$$

$$= Z[J]\cdot\Big[-\int d^4y\, \Delta(z_0-y)\,J(y) - \int d^4x\, \Delta(x-z_0)\,J(x)$$

now $\Delta(x) = \Delta(-x)$ [given]

& since x & y are dummy variables inside the integrals

$$\frac{\delta Z}{\delta J(z_0)} = -Z[J]\int d^4x\, J(x)\,\delta(z_0-X)$$

Q3 $I = \int_{-\infty}^{\infty} d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2-\xi^{-2}}$ | Assignment 4 [2019]

Spherical polar coord. $\Rightarrow$ $I = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} \frac{k^2\, dk\, e^{i\vec{k}|r|\cos\theta}}{k^2+\xi^{-2}}$

let $t = \cos\theta$

$\Rightarrow I = 2\pi \int_{-1}^{1} dt\, e^{ik|r|t} \int_0^{\infty} \frac{k^2\, dk}{k^2+\xi^{-2}}$

~~cos~~ $-\sin\theta\, d\theta = dt$

$\theta = 0 \Rightarrow \cos\theta = 1$

$\int_1^{-1} dt \Rightarrow -\int_{-1}^{1} dt$

or

$I = \frac{2\pi}{i\cancel{k}|r|} \int_0^{\infty} \frac{k\, dk}{k^2+\xi^{-2}} \left[e^{ik|r|} - e^{-ik|r|}\right]$

$= \frac{2\pi}{i|r|} \int_{-\infty}^{\infty} \frac{k\, dk\, e^{ik|r|}}{k^2+\xi^{-2}}$

let $k \to z$ (complex #)

Integral over the real axis

We can easily verify that the contribution of the infinite semicircle ~~over~~ in the upper half plane is zero [Jordon's lemma]

i.e., ~~$I = \frac{2\pi}{i|r|} \oint \frac{k\, dk\, e^{ik|r|}}{k^2+\xi^{-2}}$~~ , the pole in the upper half plane is $z = +i\xi^{-1}$

$I = \frac{2\pi}{i|r|} \oint \frac{dz\, e^{iz|r|}}{z^2+\xi^{-2}}\, z$

$= \frac{2\pi}{i|r|} \cdot 2\pi i \left[\frac{z\, e^{iz|r|}}{(z+i\xi^{-1})}\right]_{z=i\xi^{-1}} = \frac{4\pi^2 \cdot i\xi^{-1} e^{-\xi^{-1}|r|}}{2i\xi^{-1}|r|}$

$\boxed{I = \frac{2\pi^2}{|\vec{r}|} e^{-|r|\xi^{-1}}}$